Math 246A Lecture 11 Notes

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1 Singularities

1.1 Removable singularities

Theorem 1.1. *Let Ω be a domain, $\{\zeta_1, \dots \zeta_n\} \subseteq \Omega$, and suppose $f \in H(\Omega \setminus \{\zeta_1, \dots, \zeta_n\})$ such that $\lim_{z \to \zeta_j} (z - \zeta_j) f(z) = 0$. Then there exists $g \in H(\Omega)$ such that $g = f$ on $\Omega \setminus \{\zeta_1, \dots, \zeta_n\}$.*

Lemma 1.1. *Let $\Omega = \{z : |z - \zeta_j| < R\}$ and $f \in H(\Omega \setminus \{z_j\})$ such that $\lim_{z \to \zeta_j} (z - \zeta_j) f(z) = 0$. Then*

$$\int_\gamma f(z)\, dz = 0$$

for all closed $\gamma \subseteq \Omega \setminus \{\zeta_j\}$. Moreover, there exists some $F \in H(\Omega \setminus \{\zeta_j\})$ such that $F' = f$.

Proof. Let R be a rectangle in $\Omega \setminus \{\zeta_j\}$. We show that $\int_{\partial R} f(z)\, dz = 0$. If the rectangle doesn't contain ζ_j, f is holomorphic in the rectangle, so the integral is zero. If the rectangle contains ζ_j, we can reduce to a small square around ζ_j, $S = \{z : \operatorname{Re} |z - z_j| < \delta, \operatorname{Im} |z - z_j| < \delta\}$. On ∂S, $|f(z)| \sim \varepsilon / |z - \zeta_j| = \varepsilon / \delta$. The length of the perimeter of the rectangle is 4δ. So the integral goes to zero as we make the square S smaller. $\square$

Proof. Without loss of generality, $\Omega = \{z : |z - z_j| < R\}$. Let

$$h(\zeta) = \frac{f(\zeta) - f(z)}{\zeta - z}$$

for some fixed $z \in \Omega$. Then $\lim_{\zeta \to \zeta_j} h(\zeta)(\zeta - \zeta_j) = 0$. Also, $\lim_{\zeta \to z} h(\zeta)(\zeta - z) = 0$, as h is continuous. Therefore, for $r < R$ with $|\zeta_j - z| < r$,

$$\frac{1}{2\pi i} \int_{|\zeta - \zeta_j| = r} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta = 0.$$

Therefore, $|z - \zeta_j| < r < R$ implies that

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta - \zeta_j| = r} \frac{f(\zeta)}{\zeta - z}\, d\zeta := g(z).$$

This function g we have defined satisfies $g \in H(\{z : |z - \zeta| < r\})$ because we can express it $1/(\zeta - z)$ as a power series and integrate term by term. This completes the proof. $\square$

In this theorem, the ζ_j are said to be **removable singularities**.

1.2 Types of singularities

Let $f \in H(\{z : 0 < |z - z_0| < r\})$. There are three cases:

1. $\lim_{z \to z_0} (z - z_0) f(z) = 0$: This is called an **isolated singularity**. The previous theorem says that f extends to be analytic on the entire disc.

   Corollary 1.1 (Riemann)**.** *An isolated singularity of a bounded analytic function is removable.*

2. $\lim_{z \to z_0} |(z - z_0) f(z)| = \infty$: This is called a **pole** of f. Let $g = 1/f$ on $\{z : 0 < |z - z_0| < \delta\}$ for some δ where $\delta > 1$ on $0 < |z - z_0| < \delta$. So g is bounded on $0 < |z - z_0| < \delta$, so g extends to be holomorphic on $|z - z_0| < \delta$. Then $g(z) = (z - z_0)^N h(z)$, where $h(z_0) \neq 0$ and h is holomorphic on $|z - z_0| < \delta$. Then $f = 1/(z - z_0)^N (1/h)(z)$.

3. $\limsup_{z \to z_0} |f(z)| = \infty$ but $\liminf_{z \to z_0} |f(z)| < \infty$. This is called an **essential singularity**.

Example 1.1. Let $z_0 = 0$ and $f(z) = e^{1/z}$. Along the real axis, going to 0, $f(z) \to \infty$. Along the imaginary axis, going to 0, $f(z)$ spirals around the origin.

Theorem 1.2 (Casarati-Weierstrass)**.** *Let z_0 be an essential singularity of f. Then for all small $\delta > 0$, $f(\{z : 0 < |z - z_0| < \delta\})$ is dense in $\mathbb{C}$.*

Proof. Assume there exists $w_0 \in \mathbb{C}$ and $r > 0$ such that $B(w_0, r) \cap f(\{0 < |z - z_0| < \delta\}) = \varnothing$. Then $g = 1/(f - z_0)$ satisfies $|g| < 1/r$. So g has a removable singularity. Then $f = w_0 + 1/g$ has either a pole or removable singularity. $\square$

1.3 Extensions to holomorphic functions

Problem: Let $K \subseteq \mathbb{C}$ be a compact set. For which K does $f \in H(\mathbb{C} \setminus K)$ and bounded imply that f has an extension in $H(\mathbb{C})$? We have shown that if K is a finite set, this is possible. In fact, by the Baire category theorem, this works when K is countable, as well! If K is contained in the real axis, it is true iff K has measure zero. This was only solved around 15 years ago.

Theorem 1.3. *Let Ω be a domain, let $f \in H(\Omega)$, and let $B(z_0, R) \subseteq \Omega$. Then $f = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, where the series converges on $B(z_0, R)$.*

We will prove this next time.